Constrained Frobenius-Perron Operator to Analyse the Dynamics on Composed Attractors *

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In this contribution we propose a technique to analyse arbitrary invariant subsets of chaotic dynamical systems. For this purpose we introduce the constrained Frobenius-Perron operator. We demonstrate the use of this operator by determining the geometrical multifractal spectrum of invariant chaotic subsets of one-dimensional maps which are either coexisting side by side independently or are embedded in a larger set close to a crisis configuration.

1. Introduction: The Generalised Frobenius-Perron Operator and Thermodynamical Quantities

The Frobenius-Perron operator [1], H_1, of one-dimensional deterministic maps $x_{n+1}=f(x_n)$ plays an analogous role to that of the Fokker-Planck operator or the master operator in noisy systems. It is defined as

$$\hat{H}_1\,\varrho(x') = \sum_{\{x\,|\,f(x)=x'\}} \frac{\varrho(x)}{|f'(x)|}\,. \tag{1}$$

This operator transforms a function $\varrho_n(x)$ during a discrete time step into a new one $\varrho_{n+1}(x)$ while conserving the sign and the norm of the function. Thus, the Frobenius-Perron operator describes the evolution of an initial probability density $\varrho_0(x)$ toward its stationary distribution ϱ^*. (The latter is the solution of the equation $H_1\,\varrho^*(x)=\varrho^*(x)$.)

The results of the last decade have shown [2, 3] that an extension of this operator provides a powerful method to compute *thermodynamic functions*, e.g. the free energy of invariant chaotic sets in one-dimensional maps. The *generalised Frobenius-Perron operator*, $\hat{H}_\beta$, is defined [2] via

$$\hat{H}_\beta\psi(x') = \sum_{\{x\,|\,f(x)=x'\}} \frac{\psi(x)}{|f'(x)|^\beta}\,, \tag{2}$$

where β can be any real parameter. For $\beta \neq 1$ H_β does not preserve the norm and therefore the ψ functions can not be considered as probability densities.

Starting from any arbitrary smooth initial function ψ_0, the consecutive use of the generalised operator yields a sequence of functions:

$$\psi_n(y) = \hat{H}_\beta^n\,\psi_0(y) = \sum_{\{x\,|\,f^{(n)}(x)=y\}} \frac{\psi_0(x)}{|f^{(n)\prime}(x)|^\beta}\,. \tag{3}$$

In cases where only a single chaotic attractor or repellor exists, the asymptotic growth rate of these functions,

$$\lambda_1(\beta) \equiv \lim_{n\to\infty} |\psi_n(y)/\psi_0(y)|^{1/n}\,, \tag{4}$$

is *unique* for almost all initial functions and *independent of y*, and can be interpreted as the largest eigenvalue of $\hat{H}_\beta$.

The thermodynamical description of strange invariant sets of dynamical systems is based on the idea that they can be covered by a hierarchy of nested *cylinder sets* [4, 5]. Let the set

$$\{l_1^{(n)}, l_2^{(n)}, \ldots, l_{N(n)}^{(n)}\} \tag{5}$$

contain the lengths of the level-n cylinders in such a hierarchy, where $N(n)$ denotes the total number of cylinders at the nth level.

One can take the following formal "partition sum":

$$\sum_{i=1}^{N(n)} (l_i^{(n)})^\beta \sim e^{-\beta\cdot F(\beta)\cdot n}\,. \tag{6}$$

This expression defines the *free energy function* $F(\beta)$ of the chaotic set [5].

There is a nontrivial relation [2, 6]

$$\lambda_1(\beta) = e^{-\beta F(\beta)} \tag{7}$$

connecting the free energy as defined in (6) and the largest eigenvalue (4) of the generalised Frobenius-Perron operator.

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II. Coexisting Chaotic Sets

When naivly tyring to apply the above approach to maps with several coexisting chaotic invariant sets, we found that the growth rate (4) essentially depends on y as well as on the particular choice of the initial function ψ_0.

In order to give an explanation for this breakdown, let us consider the following simple situation: let us suppose that a region (R) contains a chaotic repellor [7] within the range of attraction of a disjoint chaotic attractor (A).

First, we study the case $\beta = 1$ (the conventional Frobenius-Perron operator). Any normalised smooth initial function can then be considered as a probability density function which asymptotically 'shrinks' to the attractor and approaches the density function of the natural measure. Because the natural measure is invariant, the eigenvalue $\lambda_1(1) \approx \psi_n(y)/\psi_{n-1}(y)$ will be 1 for all y points on the attractor. This, by (7), yields $F(1) = 0$. However, if y is chosen on the repellor, one finds that the density asymptotically decays with the escape rate [7] $\varkappa_R$ of the repellor: $\psi_n(y)/\psi_{n-1}(y) \approx e^{-\varkappa_R n}$, yielding a different free energy value, $F(1) = \varkappa_R$.

Let us now take the $\beta = 0$ case. If the initial function $\psi_0(x) \equiv 1$ everywhere then, according to (3), $\psi_n(y)$ simply gives the number of the order-n preimages of a point y:

$$\psi_n(y) = \sum_{\{x \mid f^{(n)}(x) = y\}} \frac{\psi_0(x)}{1} = \sum_{\{x \mid f^{(n)}(x) = y\}} 1. \tag{8}$$

There is a significant asymmetry between the dynamical roles of the two chaotic invariant sets: any point y of the attractor has preimages both on the attractor and on the repellor, while points of the repellor do not have preimages on the attractor. Thus, if y belongs to the repellor, the growth rate of the functions given by (8) is governed by the topological entropy of the repellor. If, however, y belongs to the attractor, the number of its preimages on the attractor and of those on the repellor increase with different topological entropies. The resulting growth rate thus will be dominated by the maximum of the two exponents.

The local growth rates may also change because of taking different initial functions. If $\psi_0(x)$ is chosen to be 1 on the attractor A and 0 elsewhere, then, at $\beta = 0$, (3) reads

$$\psi_n(y) = \sum_{\{x \mid f^{(n)}(x) = y\}} \frac{\psi_0(x)}{1} = \sum_{\{x \in A \mid f^{(n)}(x) = y\}} 1, \tag{9}$$

i.e., $\psi_n(y)$ counts those order-n preimages of y that lie on the attractor. Therefore, if y is on the attractor, $\psi_n(y)$ grows exponentially according to the topological entropy of the attractor, while if y lies outside the attractor, $\psi_n(y)$ remains 0 for any y since it has no preimages in (A), c.f. (9). Along similar lines it is easy to see that choosing $\psi_0(x)$ so that it is 1 on the repellor and 0 elsewhere, always yields a different growth exponent, the topological entropy of the repellor.

These examples show that in the case of coexisting disjoint invariant sets the generalised Frobenius-Perron operator method as outlined above does *not* provide us with a unique free energy function. But, as (9) shows, by applying carefully chosen constrains when selecting the y points and initial functions, it is possible to exclude or include the contribution of certain invariant set(s). This has led us to extend the concept of the Frobenius-Perron operator by involving the necessary constraints into the operator itself.

III. The Constrained Frobenius-Perron Operator

We define the *generalised Frobenius-Perron operator constrained to a* closed *set* X (constrained Frobenius-Perron operator) [10] as

$$\hat{H}_\beta^{[X]} \psi(x') = \begin{cases} \displaystyle\sum_{\{x \in X \mid f(x) = x'\}} \frac{\psi(x)}{|f'(x)|^\beta} & \text{if } x' \in X, \\ 0 & \text{otherwise}. \end{cases} \tag{10}$$

This operator acts on a space of functions with their support restricted to X. We do not require X to be contiguous: it may consist of several intervals as well.

Note that using the operator iteratively n times, the sum (11)

$$\psi_n(y) = \hat{H}_\beta^{[X]_n} \psi_0(y) = \sum_{\substack{\{x \mid f^{(n)}(x) = y \text{ and} \\ \forall m = 0, 1, \ldots, n: f^{(m)}(x) \in X\}}} \frac{\psi_0(x)}{|f^{(n)\prime}(x)|^\beta}$$

takes into account only orbits that remain within X during all the n steps. If there is an invariant subset inside X, i.e., orbits never escaping this region, then the points x in the sum above converge to this subset. This subset is typically a fractal repellor. As $n \to \infty$, (11) *reflects the properties of the invariant subset within* X only.

The essential difference between formulas (3) and (11) is that the former takes into consideration the reflux of trajectories to X, provided the dynamics allows this, while (11) describes only the contribution from the invariant set inside X without any feedback to other parts of the phase space.

When there are several disjoint invariant sets, X can be set so as to contain only the one in question. Then the growth rate of the ψ_n functions yields the first eigenvalue, $\lambda_1^{[X]}(\beta)$, of the operator $\hat{H}_\beta^{[X]}$ for (almost) all y points within X and initial functions restricted to X. The free energy $F^{[X]}(\beta)$ describing the dynamics of the investigated invariant set then follows from (3).

The possible use of the constrained Frobenius-Perron operator is, of course, not restricted to disjoint invariant sets. The "filter" set X can be chosen arbitrarily, and the corresponding free energy function will give the dynamical scaling exponents of the invariant subset within X only. For example, one can specify X so as to exclude or include paths with selected symbolic sequences. The Frobenius-Perron operator constrained to this X can then be used to analyse this artificially pruned dynamics. The invariant set within such an X (if it exists) is, of course, a part of the attractor. Therefore, the generalised Frobenius-Perron operator as defined in (10) is equally applicable to investigate *disjoint* as well as *embedded* chaotic subsets of dynamical systems. In the next section we demonstrate both cases on an example.

We would like to underline again that (11) makes possible to specify invariant subsets either *geometrically* or via the language of *symbolic dynamics.*

IV. Example: Invariant Subsets Around Crisis Induced Intermittency

As an example we choose the quadratic map

$$x_{n+1} = f(x_n) = a - x_n^2, \tag{12}$$

which has been studied intensely in the literature [8, 9]. We made investigations [10] around the critical value $a_c \equiv 1.79032749199\ldots$, at the top of the main period-3 window, where a sudden attractor enlargement [11, 12] takes place.

a) Below the crisis, within the main period-3 window $a \in [1.75\ a_c]$ of the map (12), there is a three-piece attractor (A 3). In between the three pieces of the attractor, geometrically separated from it, lies an additional invariant set, a chaotic repellor (R) with a Cantor-set-like structure. These two invariant sets are responsible for the asymptotic behaviour and the chaotic transients of the system, respectively.

When trying to apply the constrained Frobenius-Perron operator (10) to these invariant sets, it is very easy to specify the respective constraints either geometrically or by prescribing their symbolic grammar rules (for technical details see [10]). The correct specification of the constraints in (11) ensures the fast convergence and the good reliability of the numerical procedure.

Figure 1 (a) shows the free energy functions of the three-piece attractor and the coexisting repellor we have obtained at $a = 1.785$, a control parameter value where the attractor is chaotic. Apparently, the free energies of these sets are different: they characterise a relatively strong transient and a weaker asymptotic chaotic behaviour. We also show in Fig. 1 (b) the Legendre transforms of the free energies of these disjoint subsets, since they seem to be particularly useful to demonstrate the essence of the phenomenon.

The Legendre transforms of $\beta F(\beta)$ describes the geometrical multifractal properties of invariant sets [4, 5]. In order to stick to the analogy with the entropy function of phenomenological thermodynamics, we denote this quantity by $S(E)$. It can also be interpreted as the topological entropy S of trajectories with local Lyapunov exponent E. In contrast to other multifractal spectra [13], this quantity is connected to the Lebesque measure of the set and is independent of the natural measure. The graph of $S(E)$ must be a single humped convex function with its maximum giving the topological entropy of the set. For attractors, the $S(E)$ curve touches the diagonal $S = E$ at the value of the average Lyapunov exponent. For repellors, $S(E)$ is shifted to the right from the diagonal by an amount of the escape rate $\varkappa$. The average Lyapunov exponent is, in general, that value of E where the graph has a unit slope [5].

Figure 1 (b) shows that the repellor is more chaotic than the coexisting attractor because its $S(E)$ spectrum is much further away to the right than that of the attractor (i.e., it has strictly larger local Lyapunov exponents) and, also, has a greater maximum (i.e., topological entropy).

b) Beyond the crisis ($a > a_c$) there is only a single one-piece enlarged attractor (A). The free energy of this attractor can be obtained by using the unconstrained Frobenius-Perron operator (2) or, equivalently, by taking the trivial "constraint" $X = A$ in the constrained one (10, 11).

However, it is also possible to apply the same constraints that were used in the precritical case to specify

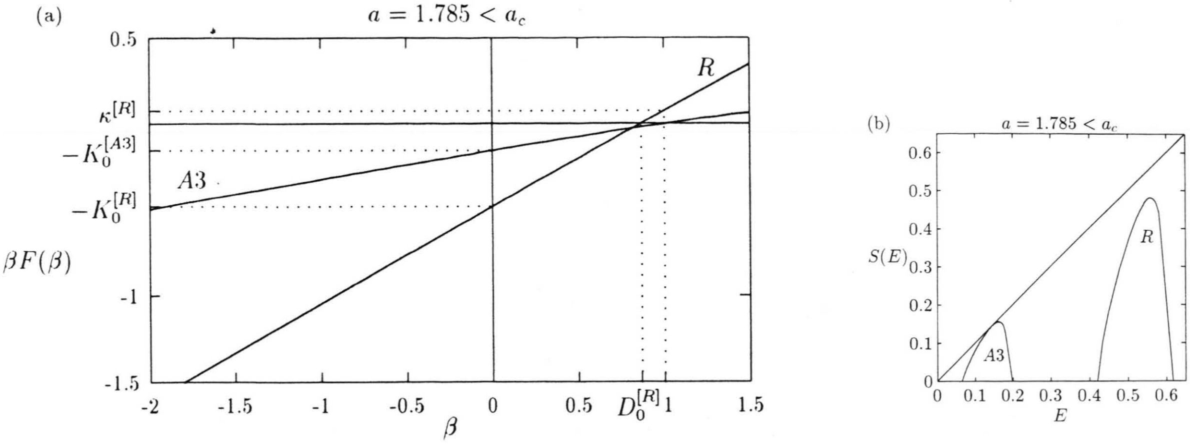

Fig. 1. The geometrical multifractal spectrum of coexisting chaotic sets at $a = 1.785$ in the precritical regime of the quadratic map (12). (a) The free energies of the three-piece attractor (A 3) and of the repellor (R) were obtained by the constrained Frobenius-Perron operator method. The topological entropy K_0 and the fractal dimension D_0 of these sets are given by the intersection of their $\beta F(\beta)$ functions with the vertical and horizontal axes, respectively. The escape rate $\varkappa$ is the value of the function at $\beta = 1$. (b) The $S(E)$ functions of the invariant sets obtained by numerical Legendre transformation from the corresponding free energies.

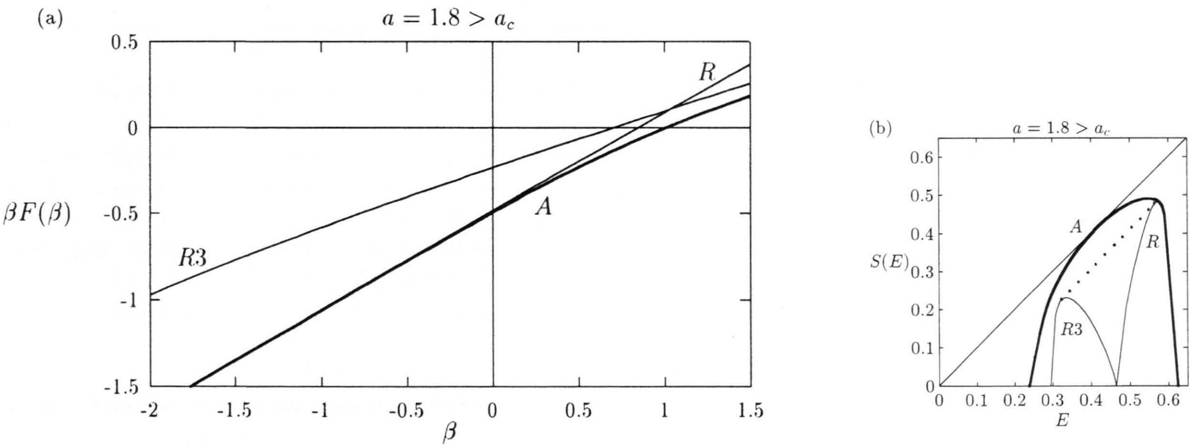

Fig. 2. The geometrical multifractal spectrum of chaotic sets at $a = 1.8$ in the postcritical regime. (a) The free energy of the entire attractor (A, solid line) was obtained by the generalised Frobenius-Perron operator method, while the free energies of the two repellor components R 3 and R, by the constrained Frobenius-Perron operator method. (b) The $S(E)$ function of the entire attractor (A) describing the asymptotic chaotic dynamics is somewhat above the convex hull (dotted line) of the two repellors. The latter spectra reflect the short lifetime behaviour of the two intermittent chaotic phases. The typical lifetimes of these chaotic transients are given by the inverse escape rates of the corresponding repellors.

invariant subsets. The one which is defined the same way as the repellor in the precritical case inherits exactly the same topological structure, therefore we keep denoting this subset by (R). The other one with the same constraints as the previous three-piece attractor consists now of those orbits that never escape from the three bands of the former attractor. This set will be called hereafter the three-piece repellor (R3) as being the remnant of the precritical three-piece attractor (A 3).

Both of these invariant subsets are repellors now with zero Lebesque measure and a Cantor-set-like structure. They are completely *embedded* in the attractor and they represent two sorts of *transient* chaotic motion which are present *within the asymptotic motion* on the attractor. In fact these embedded repellors are responsible for the crisis induced intermittent behaviour [12] of the system in the vicinity of the critical value a_c. The dynamics on the whole attractor can then be considered as intermittent switchings between these two sorts of chaotic transients.

Figure 2 (a) shows the free energies at $a = 1.8$, somewhat beyond the crisis situation. The free energy curve of the enlarged attractor runs below the curves belonging to the embedded repellors. This is a consequence of the fact that the largest eigenvalue of the full Frobenius-Perron operator (2) limitates the eigenvalues of the constrained Frobenius-Perron operators (10). This, according to (7), implies that the free energy of an attractor gives a lower limit for the free-energies of its components.

We can take advantage of this property by reversing this argument: it seems to be worthwhile approaching the multifractal spectrum of the attractor with those of its embedded repellor components. The attractor can have a complex structure with weak couplings which, as it is the case close to critical situations, often results in bad convergence properties in (11). On the other hand, just in such situations do the remnants of the disjoint precritical invariant sets have simple grammar providing fast computation of their free energies free of numerical complications. The free energy of the whole attractor then can be *approximated* by that of its embedded nonattractive components. What is more, such an approach provides us with some information on the internal structure of the attractor and on the characteristics of the asymptotic behaviour; for example the escape rates from the individual repelling components determine the characteristic lifetime and frequency of bursts in the case of crisis induced intermittency.

By considering the $S(E)$ functions corresponding to the free energies on Figure 2 (b), it is conspicuous that the spectrum characterising the whole attractor runs indeed somewhat above the common convex envelope of the curves belonging to the two repellors. This fact shows convincingly that the multifractal spectra of the embedded repellor components can be used as a sort of frame, or backbone, to approach the spectrum of the whole attractor [14].

V. Remarks and Outlook

In the above example we showed that by applying the Frobenius-Perron operator with suitably chosen constraints one can get rid of the initial function- and y-dependence, and can approach the multifractal spectrum of a large invariant set with that of its components. This approach can be improved by using more repellor components. Thus, the constrained Frobenius-Perron operator provides a useful tool for the investigation of *multitransient chaos* [15].

We would like to mention that the fact that the $S(E)$ functions of the disjoint components do not lap over is a consequence of the specific map, and this is not necessarily true for other systems. Specially, if the system has certain symmetries, like in cases of symmetry recovering attractor mergings [9], different invariant sets may have identical free energy functions.

It seems that the idea of introducing constraints into the master operator of dynamical systems to obtain the thermodynamic potential of chaotic subsets can be extended to higher dimensional maps as well. In order to that, appropriate generalisations should be made in the definition of thermodynamical quantities and of operators like, e.g., the one proposed in [16].

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